Nonadditive quantum error-correcting code

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We report the first nonadditive quantum error-correcting code, namely, a $((9,12,3))$ code which is a 12-dimensional subspace within a 9-qubit Hilbert space, that outperforms the optimal stabilizer code of the same length by encoding more levels while correcting arbitrary single-qubit errors.

The quantum error-correcting code (QECC) [1, 2, 3, 4] provides an active way of protecting our quantum data from decohering. Almost all the QECCs constructed so far are stabilizer codes [5, 6, 7], codes that have the structure of an eigenspace of an Abelian group generated by mulitilocal Pauli operators. Codes without such a structure are called nonadditive codes. The first nonadditive code [8, 9] that outperforms the stabilizer codes is the $((5,6,2))$ code, a 5-qubit code encoding 6 levels capable of correcting single-qubit *erasure*, i.e., a code of distance 2. Recently a family of distance 2 nonadditive codes with a higher encoding rate has been constructed [10]. Though some nonadditive error-correcting codes had been constructed [11, 12], the question of whether the nonadditive error-correcting codes with a distance larger than 2 can encode more levels than the corresponding stabilizer codes remains open.

In this Letter we report the first nonadditive code of distance 3 that beats the corresponding stabilizer code: a nonadditive $((9,12,3))$ code that is a 12-dimensional subspace in a 9-qubit Hilbert space against arbitrary single-qubit errors. In comparison, the best stabilizer code $[[9,3,3]]$ of the same length can encode only 3 logical qubits, i.e., an 8-dimensional subspace [7].

Our new code is most conveniently formulated in terms of graph states [13, 14]. Let $G=(V,\Gamma)$ be an undirected simple graph with $|V|=n$ vertices and Γ, called as the *adjacency matrix* of the graph, is an $n\times n$ symmetric matrix with vanishing diagonal entries and $\Gamma_{ab}=1$ if vertices a,b are connected and $\Gamma_{ab}=0$ otherwise. Consider a system of n qubits labeled by V and denote by $\mathcal{X}_a,\mathcal{Y}_a$, and $\mathcal{Z}_a$ three Pauli operators acting on qubit $a\in V$. The *graph state* associated with graph G reads

$$|G\rangle=\prod_{\Gamma_{ab}=1}\mathcal{U}_{ab}\,|+\rangle_x^V=\frac{1}{\sqrt{2^n}}\sum_{\vec{\mu}=\mathbf{0}}^{\mathbf{1}}(-1)^{\frac{1}{2}\vec{\mu}\cdot\Gamma\cdot\vec{\mu}}\,|\vec{\mu}\rangle_z\,, \quad (1)$$

where $|\vec{\mu}\rangle_z$ is the common eigenstates of $\{\mathcal{Z}_a\}_{a\in V}$ with $(-1)^{\mu_a}$ as eigenvalues, $|+\rangle_x^V$ denotes the simultaneous $+1$ eigenstate of $\{\mathcal{X}_a\}_{a\in V}$, and $\mathcal{U}_{ab}=(1+\mathcal{Z}_a+\mathcal{Z}_b-\mathcal{Z}_a\mathcal{Z}_b)/2$ is the controlled-phase operation between qubit a and b. The graph state is also the unique simultaneous $+1$ eigenstate of n vertex stabilizers $\mathcal{G}_a=\mathcal{X}_a\mathcal{Z}_{N_a}$ with $a\in V$ where N_a is the neighborhood of a and we denote by $\mathcal{Z}_U=\prod_{a\in U}\mathcal{Z}_a$ for a subset of vertices $U\subseteq V$.

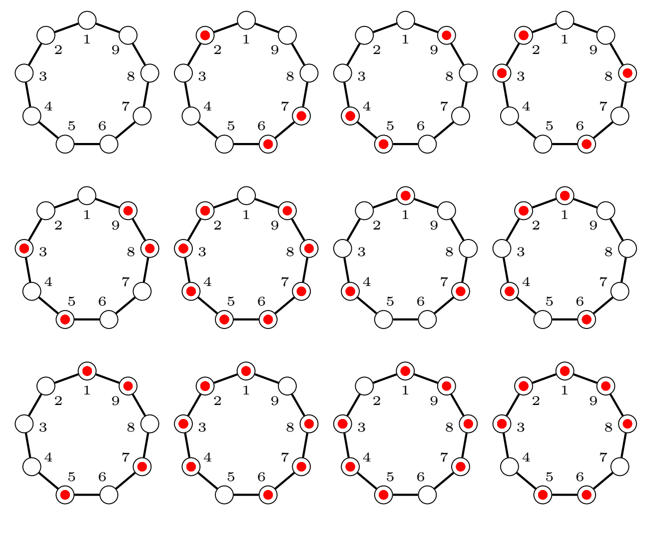

FIG. 1: (Color online) Twelve graph-state bases on the loop graph L_9 for the $((9,12,3))$ code $\mathbb{D}$. Each graph represents a graph state that is the unique common eigenstate of the vertex stabilizers $\{\mathcal{G}_a\}$ with eigenvalue $+1$ if a is uncolored and -1 if the vertex is red-colored.

We consider in what follows the loop graph L_9 on 9 vertices which are labeled by integers from 1 to 9. Its adjacency matrix has nonvanishing entries $\Gamma_{aa_\pm}=1$ $(1\le a\le 9)$ only where $a_\pm=a\pm1$ with identifications $9_+=1$ and $1_-=9$. The corresponding graph state is denoted as $|L_9\rangle$. We claim that the 12-dimensional subspace $\mathbb{D}$ spanned by the states $\{\mathcal{Z}_{V_i}|L_9\rangle\}_{i=1}^{12}$ where

$$\begin{gathered}V_1=\emptyset,\ V_2=\{2,6,7\},\ V_3=\{4,5,9\},\ V_4=\{2,3,6,8\}\\ V_5=\{3,5,8,9\},\ V_6=\{2,3,4,5,6,7,8,9\}\\ V_7=\{1,4,7\},\ V_8=\{1,2,4,6\},\ V_9=\{1,5,7,9\}\\ V_{10}=\{1,2,3,4,6,7,8\},\ V_{11}=\{1,3,4,5,7,8,9\}\\ V_{12}=\{1,2,3,5,6,8,9\},\end{gathered} \quad (2)$$

as shown in Fig.1, is a $((9,12,3))$ code. Obviously these 12 states are mutually orthogonal since $V_i's$ are distinct and $\langle G|\,\mathcal{Z}_V\,|G\rangle=\delta_{V,\emptyset}$ holds true for any graph state. To prove that the code is of distance 3, i.e., capable of correcting single-qubit errors, we have only to demonstrate that each one of 3×9 single-qubit errors and 9×36 two-qubit errors $\mathcal{E}$ will bring $\mathbb{D}$ into its orthogonal complement [4, 13], i.e.,

$$\langle L_9|\,\mathcal{Z}_{V_i}\mathcal{E}\mathcal{Z}_{V_j}\,|L_9\rangle=0,\quad (1\le i,j\le 12). \quad (3)$$

Since all the bases of $\mathbb{D}$ given above are the common eigenstates of the vertex stabilizers $\{\mathcal{G}_a = \mathcal{Z}_{a-}\mathcal{X}_a\mathcal{Z}_{a+}\}_{a=1}^9$, a bit flip error $\mathcal{X}_a$ on these bases is equivalent to a phase flip error $\mathcal{Z}_{N_a}$ on qubits in its neighborhood, e.g., $N_a = \{a_+, a_-\}$ in L_9, upto an unimportant phase factor. And a $\mathcal{Y}_a$ error can be equivalently replaced by a phase flip error $\mathcal{Z}_a\mathcal{Z}_{N_a}$ on quibts a, a_+, and a_-. As a result every single-qubit error is equivalent to one of the following phase flip errors $\{\mathcal{Z}_a, \mathcal{Z}_{N_a}, \mathcal{Z}_a\mathcal{Z}_{N_a}\}$ for $1 \leq a \leq 9$ and every two-qubit error is equivalent to one of the following phase flip errors

$$\begin{array}{ccc} \mathcal{Z}_a\mathcal{Z}_b, & \mathcal{Z}_{N_a}\mathcal{Z}_b, & \mathcal{Z}_{N_a}\mathcal{Z}_a\mathcal{Z}_b, \\ \mathcal{Z}_a\mathcal{Z}_{N_b}, & \mathcal{Z}_{N_a}\mathcal{Z}_{N_b}, & \mathcal{Z}_a\mathcal{Z}_{N_a}\mathcal{Z}_{N_b}, \\ \mathcal{Z}_a\mathcal{Z}_b\mathcal{Z}_{N_b}, & \mathcal{Z}_{N_a}\mathcal{Z}_{N_b}\mathcal{Z}_b, & \mathcal{Z}_a\mathcal{Z}_b\mathcal{Z}_{N_a}\mathcal{Z}_{N_b}, \end{array} \tag{4}$$

with $1 \leq a, b \leq 9$. To summarize, for a loop graph, every single-qubit or two-qubit error is equivalent to one of following 6 patterns of phase flip errors

$$\begin{array}{rl} \mathrm{I}: & \mathcal{Z}_a, \\ \mathrm{II}: & \mathcal{Z}_a\mathcal{Z}_b, \\ \mathrm{III}: & \mathcal{Z}_{a-}\mathcal{Z}_b\mathcal{Z}_{a+},\ \mathcal{Z}_{a\pm}\mathcal{Z}_a\mathcal{Z}_{a\pm3}, \\ \mathrm{IV}: & \mathcal{Z}_{a-}\mathcal{Z}_{a+}\mathcal{Z}_{b-}\mathcal{Z}_{b+},\ \mathcal{Z}_{a-}\mathcal{Z}_a\mathcal{Z}_{a+}\mathcal{Z}_b, \\ & \mathcal{Z}_{a-}\mathcal{Z}_{a-2}\mathcal{Z}_{a+}\mathcal{Z}_{a+2}, \\ \mathrm{V}: & \mathcal{Z}_{a-}\mathcal{Z}_a\mathcal{Z}_{a+}\mathcal{Z}_{b-}\mathcal{Z}_{b+}, \\ \mathrm{VI}: & \mathcal{Z}_{a-}\mathcal{Z}_a\mathcal{Z}_{a+}\mathcal{Z}_{b-}\mathcal{Z}_b\mathcal{Z}_{b+}, \end{array} \tag{5}$$

where a, b are suitably chosen so that error patterns I, II, III, IV, V, VI are phase flip errors on 1 qubit to 6 qubits respectively. It is clear that phase flip errors on more than 6 qubits cannot be caused by any single-qubit or two-qubit error.

As an immediate consequence, Eq.(3) is equivalent to saying that *none* of the transition operators $\mathcal{Z}_{V_i}\mathcal{Z}_{V_j}$ ($1 \leq i < j \leq 12$) between each pair of bases of $\mathbb{D}$ belongs to any one of 6 error patterns listed in Eq.(5). Because $\mathcal{Z}_{V_k}\mathcal{Z}_{V_7} = \mathcal{Z}_{V_{k+6}}$ it is enough to examine the following 31 different transition operators

$$\begin{array}{lllllll} \mathcal{Z}_{147}, & \mathcal{Z}_{126}, & \mathcal{Z}_{1246}, & \mathcal{Z}_{2368}, & \mathcal{Z}_{12569}, & \mathcal{Z}_{1234678}, & \mathcal{Z}_{12345689}, \\ & \mathcal{Z}_{159}, & \mathcal{Z}_{1348}, & \mathcal{Z}_{2569}, & \mathcal{Z}_{23678}, & \mathcal{Z}_{1235689}, & \mathcal{Z}_{12356789}, \\ & \mathcal{Z}_{267}, & \mathcal{Z}_{1378}, & \mathcal{Z}_{3589}, & \mathcal{Z}_{34589}, & \mathcal{Z}_{1245679}, & \mathcal{Z}_{23456789}, \\ & \mathcal{Z}_{348}, & \mathcal{Z}_{1579}, & & \mathcal{Z}_{123468}, & \mathcal{Z}_{1345789}, & \\ & \mathcal{Z}_{378}, & \mathcal{Z}_{2467}, & & \mathcal{Z}_{135789}, & \mathcal{Z}_{2345689}, & \\ & \mathcal{Z}_{459}, & \mathcal{Z}_{4579}, & & \mathcal{Z}_{245679}, & \mathcal{Z}_{2356789}. & \end{array}$$

obtained from $\mathcal{Z}_{V_7}$ and $\{\mathcal{Z}_{V_i}\mathcal{Z}_{V_j}, \mathcal{Z}_{V_7}\mathcal{Z}_{V_i}\mathcal{Z}_{V_j} | 1 \leq i < j \leq 6\}$. It is easy to check that phase flip errors on 5 or more qubits in the above table do not belong to any one of the error patterns in Eq.(5). Because of the symmetry of the loop graph L_9, one needs only to check that $\mathcal{Z}_{126}$, $\mathcal{Z}_{147}$,$\mathcal{Z}_{1246}$, and $\mathcal{Z}_{2368}$ do not belong to any one of the error patterns in Eq.(5), which are easy tasks to perform. In this way we have demonstrated that $\mathbb{D}$ is a ((9,12,3)) code, which is obviously nonadditive.

As to the projector of the code $\mathbb{D}$, we notice that there are 3 local stabilizers of the code $\mathbb{D}$, namely, $\mathcal{G}_{38}, \mathcal{G}_{62}$, and $\mathcal{G}_{95}$, where we have denoted $\mathcal{G}_U = \prod_{v\in U}\mathcal{G}_v$ for a subset of vertices U. By denoting

$$\begin{aligned} \mathcal{A} \ = \ & \mathcal{G}_{14}\Big(1 - \mathcal{G}_{36} + \mathcal{G}_{39} - \mathcal{G}_{69} + 2\mathcal{G}_{369} + 2\mathcal{G}_9\Big) \\ & + \mathcal{G}_{17}\Big(1 - \mathcal{G}_{39} + \mathcal{G}_{36} - \mathcal{G}_{69} + 2\mathcal{G}_{369} + 2\mathcal{G}_6\Big), \end{aligned} \tag{6}$$

we can write down the projector of the code $\mathbb{D}$ as

$$\mathcal{P} = \frac{1}{2^{10}}(1 + \mathcal{G}_{38})(1 + \mathcal{G}_{62})(1 + \mathcal{G}_{95})\mathcal{A}(\mathcal{A} + 8), \tag{7}$$

from which the weight enumerator [15, 16] of the code $\mathbb{D}$ can be readily obtained

$$2^9 \times 12 \times \left(\frac{3u^9}{128} + \frac{u^5v^4}{64} + \frac{u^3v^6}{4} + \frac{u^2v^7}{2} + \frac{27uv^8}{128}\right). \tag{8}$$

Here the coefficients of $u^{9-d}v^d$ is given by $\sum \mathrm{Tr}^2(\mathcal{P}\mathcal{E}_d)$ with the summation taken over all (Hermitian) errors acting nontrivially on d qubits.

To conclude, we have provided the first evidence that nonadditive error-correcting codes can perform better than the stabilizer codes. Since the bases of our code are all graph states, they can be easily be prepared from a product state by using controlled phase operation and local unitary operations as shown in Eq.(1).

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